Trinity College

Semester One Examination, 2017

Question/Answer booklet

MATHEMATICS METHODS UNITS 3,4

Section Two: Calculator-assumed

SOLUTIONS

Student Number: In figures

In words ______________________________

Your name ______________________________

Time allowed for this section

Reading time before commencing work: ten minutes
Working time: one hundred minutes

Materials required/recommended for this section

To be provided by the supervisor
This Question/Answer booklet
Formula sheet (retained from Section One)

To be provided by the candidate
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators approved for use in this examination

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	8	8	50	54	35
Section Two: Calculator-assumed	11	11	100	98	65
				Total	100

Instructions to candidates

1. The rules for the conduct of Trinity College examinations are detailed in the *Instructions to Candidates* distributed to students prior to the examinations. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet.

3. You must be careful to confine your response to the specific question asked and to follow any instructions that are specified to a particular question.

4. Additional working space pages at the end of this Question/Answer booklet are for planning or continuing an answer. If you use these pages, indicate at the original answer, the page number it is planned/continued on and write the question number being planned/continued on the additional working space page.

5. Show all your working clearly. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

6. It is recommended that you do not use pencil, except in diagrams.

7. The Formula sheet is not to be handed in with your Question/Answer booklet.

Section Two: Calculator-assumed **65% (98 Marks)**

This section has **eleven (11)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 100 minutes.

Question 9 **(7 marks)**

The voltage between the plates of a discharging capacitor can be modelled by the function $V(t) = 14e^{kt}$, where V is the voltage in volts, t is the time in seconds and k is a constant.

It was observed that after three minutes the voltage between the plates had decreased to 0.6 volts.

(a) State the initial voltage between the plates. (1 mark)

Solution
$V_0 = 14$ volts
Specific behaviours
✓ states value (units not required)

(b) Determine the value of k. (2 marks)

Solution
$0.6 = 14e^{180k}$ $k = -0.0175$
Specific behaviours
✓ writes equation ✓ solves, rounding to 3sf

(c) How long did it take for the initial voltage to halve? (2 marks)

Solution
$0.5 = e^{-0.0175t}$ $t = 39.6$ s
Specific behaviours
✓ writes equation ✓ solves, rounding to 3sf

(d) At what rate was the voltage decreasing at the instant it reached 8 volts? (2 marks)

Solution
$V'(t) = kV$ $= -0.0175 \times 8 = -0.14$ Decreasing at 0.14 volts/s
Specific behaviours
✓ uses rate of change ✓ states decrease, dropping negative sign

Question 10 **(11 marks)**

The gradient function of f is given by $f'(x) = 12x^3 - 24x^2$.

(a) Show that the graph of $y = f(x)$ has two stationary points. (2 marks)

Solution
Require $f'(x) = 0$ $12x^2(x-2) = 0$ $x = 0, x = 2$ Hence two stationary points
Specific behaviours
✓ equates derivative to zero and factorises ✓ shows two solutions and concludes two stationary points

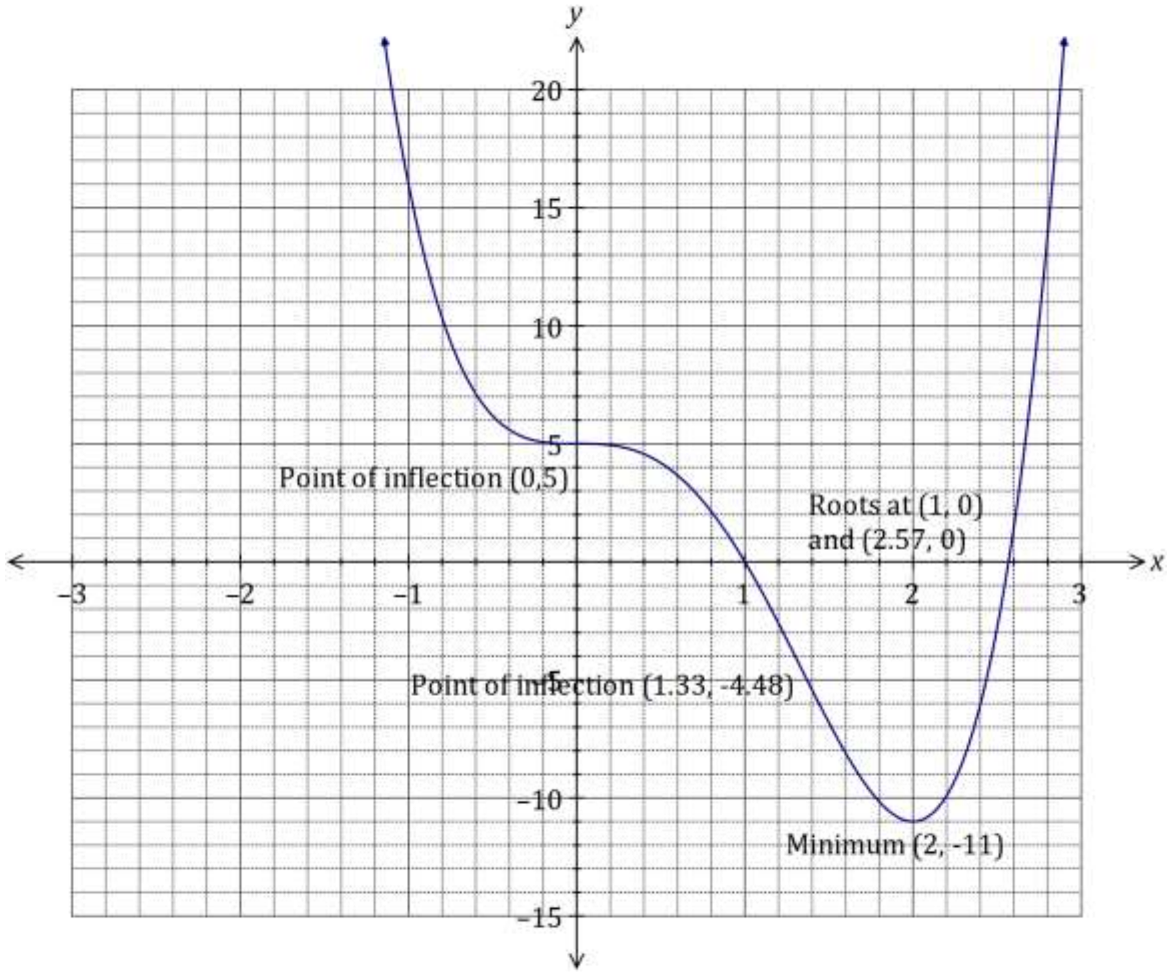
(b) Determine the interval(s) for which the graph of the function is concave upward. (3 marks)

Solution
$f''(x) = 36x^2 - 48x$ $f''(x) > 0 \Rightarrow x < 0, x > \frac{4}{3}$
Specific behaviours
✓ shows condition for concave upwards ✓ uses second derivative ✓ states intervals

(c) Given that the graph of $y = f(x)$ passes through $(1, 0)$, determine $f(x)$. (2 marks)

Solution
$f(x) = \int f'(x)\,dx = 3x^4 - 8x^3 + c$ $f(1) = 0 \Rightarrow c = 5$ $f(x) = 3x^4 - 8x^3 + 5$
Specific behaviours
✓ integrates $f'(x)$ ✓ determines constant

(d) Sketch the graph of $y = f(x)$, indicating all key features. (4 marks)



Solution
See graph
Specific behaviours
✓ minimum ✓ roots ✓ points of inflection ✓ smooth curve

Question 11 **(7 marks)**

(a) Four random variables W and Z are defined below. State, with A reason, whether the distribution of the random variable is Bernoulli, binomial, uniform or none of these.

The dice referred to is a cube with faces numbered with the integers 1, 2, 3, 4, 5 and 6.

(i) W is the number of throws of a dice until a six is scored. (2 marks)

Solution
Neither - distribution is geometric
Specific behaviours
✓✓ answer with valid reason

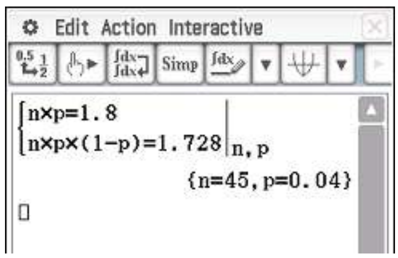
(iv) Z is the total of the scores when two dice are thrown. (2 marks)

Solution
Neither - distribution is triangular
Specific behaviours
✓✓ answer with valid reason

(b) Pegs produced by a manufacturer are known to be defective with probability p, independently of each other. The pegs are sold in bags of n for \$4.95. The random variable X is the number of faulty pegs in a bag.

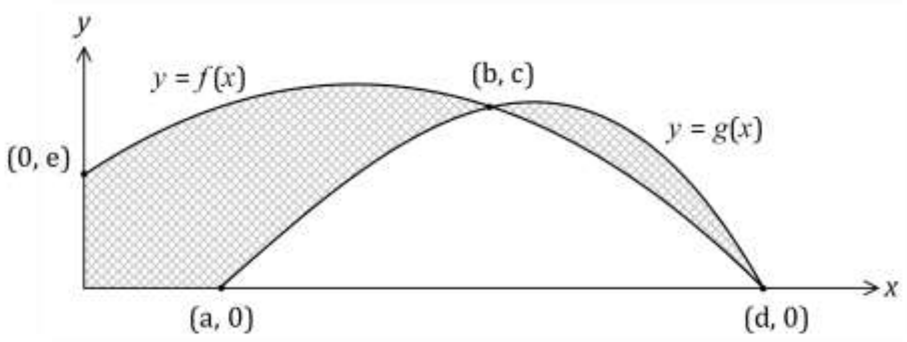
If $E(X) = 1.8$ and $Var(X) = 1.728$, determine n and p. (3 marks)

Solution
$np = 1.8, np(1-p) = 1.728$ $\therefore 1 - p = \frac{1.728}{1.8} = 0.96$ $p = 0.04$ $n = \frac{1.8}{0.04} = 45$
Specific behaviours
✓ writes equations for mean and variance ✓ solves for p ✓ solves for n



Question 12 **(7 marks)**

The graphs of the functions f and g are shown below, intersecting at the points (b, c) and $(d, 0)$.



(a) Using definite integrals, write an expression for the area of the shaded region. (3 marks)

Solution
$\text{Area} = \int_0^b f(x)\,dx - \int_a^b g(x)\,dx + \int_b^d (g(x) - f(x))\,dx$
Specific behaviours
✓ area from $x = 0$ to $x = b$ ✓ area from $x = b$ to $x = d$ ✓ uses correct notation throughout

(b) Evaluate the area of the shaded region when $f(x) = 15 + 12x - 3x^2$ and $g(x) = -x^3 + 3x^2 + 13x - 15$.

(4 marks)

Solution
$a = 1, b = 3, d = 5$ $\int_0^3 f(x)\,dx - \int_1^3 g(x)\,dx = 72 - 28 = 44$ $\int_3^5 (g(x) - f(x))\,dx = 8$ $\text{Total area} = 44 + 8 = 52 \text{ sq units}$
Specific behaviours
✓ determines values of a, b and d ✓ area from $x = 0$ to $x = b$ ✓ area from $x = b$ to $x = d$ ✓ correct area

Question 13 **(9 marks)**

75% of the avocados produced by a farm are known to be first grade, the rest being second grade. Trays of 24 avocados are filled at random in a packing shed and sent to market.

Let the random variable X be the number of first grade avocados in a single tray.

(a) Explain why X is a discrete random variable, and identify its probability distribution. (2 marks)

Solution
X is a DRV as it can only take integer values from 0 to 24. X follows a binomial distribution: $X \sim B(24, 0.75)$
Specific behaviours
✓ explanation using discrete values ✓ identifies binomial, with parameters

(b) Calculate the mean and standard deviation of X. (2 marks)

Solution
$\bar{X} = 24 \times 0.75 = 18$ $\sigma_x = \sqrt{18 \times 0.25} = \dfrac{3\sqrt{2}}{2} \approx 2.12$
Specific behaviours
✓ mean, ✓ standard deviation

(c) Determine the probability that a randomly chosen tray contains

(i) 18 first grade avocados. (1 mark)

Solution
$P(X = 18) = 0.1853$
Specific behaviours
✓ probability

(ii) more than 15 but less than 20 first grade avocados. (2 marks)

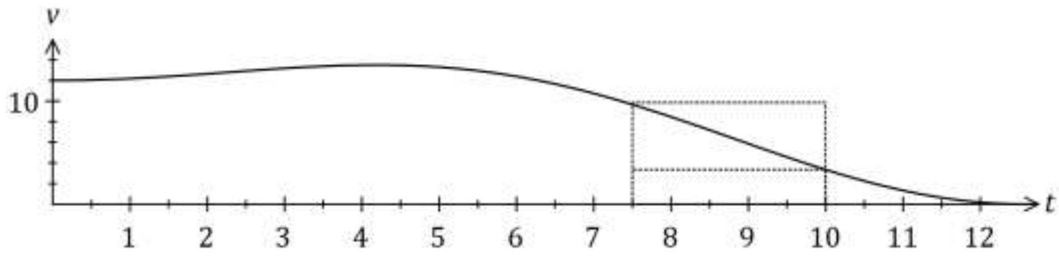
Solution
$P(16 \leq X \leq 19) = 0.6320$
Specific behaviours
✓ uses correct bounds ✓ probability

(d) In a random sample of 1000 trays, how many trays are likely to have fewer first grade than second grade avocados. (2 marks)

Solution
$P(X \leq 11) = 0.0021$ $0.0021 \times 1000 \approx 2 \text{ trays}$
Specific behaviours
✓ identifies upper bound and calculates probability ✓ calculates whole number of trays

Question 14 **(8 marks)**

The speed, in metres per second, of a car approaching a stop sign is shown in the graph below and can be modelled by the equation $v(t) = 6(1 + \cos(0.25t) + \sin^2(0.25t))$, where t represents the time in seconds.



The area under the curve for any time interval represents the distance travelled by the car.

(a) Complete the table below, rounding to two decimal places. (2 marks)

t	0	2.5	5	7.5	10
$v(t)$	12.00	12.92	13.30	9.66	3.34

Solution
See table
Specific behaviours
✓ values, ✓ rounding

(b) Complete the following table and hence estimate the distance travelled by the car during the first ten seconds by calculating the mean of the sums of the inscribed areas and the circumscribed areas, using four rectangles of width 2.5 seconds.
(The rectangles for the 7.5 to 10 second interval are shown on the graph.) (5 marks)

Interval	$0 - 2.5$	$2.5 - 5$	$5 - 7.5$	$7.5 - 10$
Inscribed area	30.0	32.3	24.15	8.35
Circumscribed area	32.3	33.25	33.25	24.15

Solution
See table (*may have slightly different values if using exact values of $v(t)$ rather than those from (a)*) $\sum \text{Inscribed} = 94.8, \sum \text{Circumscribed} = 122.95$ $\text{Estimate} = \dfrac{94.8 + 122.95}{2} \approx 108.9 \text{ m}$
Specific behaviours
✓ values 1st col, ✓ values 2nd col, ✓ values 3rd col ✓ sums ✓ estimate that rounds to 109

(c) Suggest one change to the above procedure to improve the accuracy of the estimate. (1 mark)

Solution
Use a larger number of thinner rectangles.
Specific behaviours
✓ valid suggestion

Question 15 **(10 marks)**

A slot machine is programmed to operate at random, making various payouts after patrons pay \$2 and press a start button. The random variable X is the amount of the payout, in dollars, in one play of the machine. Each payout can be assumed to be independent of other payouts.

The probability, P, that the machine makes a certain payout, x, is shown in the table below.

Payout (\$) x	0	1	2	5	10	20	50	100
Probability $P(X = x)$	0.25	0.45	0.2125	0.0625	0.0125	0.005	0.005	0.0025

(a) Determine the probability that

(i) in one play of the machine, a payout of more than \$1 is made. (1 mark)

Solution
$P(X > 1) = 1 - (0.25 + 0.45) = 0.3$
Specific behaviours
✓ states probability

(ii) in ten plays of the machine, it makes a payout of \$5 no more than once. (2 marks)

Solution
$Y \sim B(10, 0.0625)$ $P(Y \leq 1) = 0.8741$
Specific behaviours
✓ indicates binomial distribution ✓ calculates probability

(iii) in five plays of the machine, the second payout of \$1 occurs on the fifth play. (3 marks)

Solution
First payout in one of four plays: $W \sim B(4, 0.45)$ $P(W = 1) = 0.2995$ Second payout: $P = 0.2995 \times 0.45 = 0.1348$
Specific behaviours
✓ uses first and second event ✓ calculates P for first event ✓ calculates P for both events

(b) Calculate the mean and standard deviation of X. (2 marks)

Solution
$\bar{X} = 1.9125, \sigma_X = 6.321$
Specific behaviours
✓ mean ✓ sd

(c) In the long run, what percentage of the player's money is returned to them? (2 marks)

Solution
$\dfrac{1.9125}{2} \times 100 = 95.625\%$
Specific behaviours
✓ uses mean and payment ✓ calculates percentage

Question 16 **(12 marks)**

Particle P leaves point A at time $t = 0$ seconds and moves in a straight line with acceleration given by

$$a = \frac{16}{(2t+1)^3}\ \mathrm{ms}^{-2}.$$

Particle P has an initial velocity of $-3\ \mathrm{ms}^{-1}$ and point A has a displacement of 4 metres from the origin.

(a) Calculate the initial acceleration of P. (1 mark)

Solution
$a(0) = 16\ \mathrm{ms}^{-2}$
Specific behaviours
✓ correct value

(b) Is P ever stationary? If your answer is yes, determine the time(s) when this happens. If your answer is no, explain why. (3 marks)

Solution
$v = \int a\,dt = \frac{-4}{(2t+1)^2} + c$ $t = 0, v = -3 \Rightarrow c = 1$ $v = \frac{-4}{(2t+1)^2} + 1$ $v = 0 \Rightarrow t = 0.5\ \mathrm{s}$ YES. P is stationary when $t = 0.5$ s
Specific behaviours
✓ integrates to find velocity ✓ correct constant ✓ solves for zero

(c) Calculate the displacement of P when $t = 12$ seconds. (2 marks)

Solution
$\Delta x = \int_0^{12} v\,dt = 10.08$ $x(12) = 4 + 10.08 = 14.08\ \mathrm{m}$
Specific behaviours
✓ integrates to find change in displacement ✓ calculates actual displacement

(d) Calculate the change of displacement of P during the third second. (2 marks)

Solution
$\Delta x = \int_2^3 v\,dt = \frac{31}{35} \approx 0.886 \text{ m}$
Specific behaviours
✓ uses correct bounds ✓ integrates to find change in displacement

(e) Determine the maximum speed of P during the first three seconds and the time when this occurs. (2 marks)

Solution
Observe $\lvert v\rvert$ decreases then increases: $\lvert v(0)\rvert = 3, \lvert v(3)\rvert \approx 0.92$ Hence maximum speed is 3 ms^{-1}.
Specific behaviours
✓ examines v at endpoints ✓ determines maximum speed

(f) Calculate the total distance travelled by P during the first three seconds. (2 marks)

Solution
$d = \int_0^3 \lvert v\rvert\,dt \text{ or } d = -\int_0^{0.5} v\,dt + \int_{0.5}^3 v\,dt$ $d = \frac{16}{7} \approx 2.286 \text{ m}$
Specific behaviours
✓ uses integral(s) to determine distance ✓ evaluates distance

Question 17 **(10 marks)**

Let the random variable X be the number of vowels in a random selection of four letters from those in the word LOGARITHM, with no letter to be chosen more than once.

(a) Complete the probability distribution of X below. (1 mark)

x	0	1	2	3
$P(X = x)$	$\frac{5}{42}$	$\frac{10}{21}$	$\frac{5}{14}$	$\frac{1}{21}$

Solution
$1 - \left(\frac{5}{42} + \frac{10}{21} + \frac{1}{21}\right) = \frac{5}{14}$
Specific behaviours
✓ uses sum of probabilities

(b) Show how the probability for $P(X = 1)$ was calculated. (2 marks)

Solution
$P(X = 1) = \frac{\binom{3}{1} \times \binom{6}{3}}{\binom{9}{4}} = \frac{3 \times 20}{126} = \frac{10}{21}$
Specific behaviours
✓ uses combinations for numerator ✓ uses combinations for denominator and simplifies

(c) Determine $P(X \geq 1 | X \leq 2)$. (2 marks)

Solution
$P = \frac{\frac{10}{21} + \frac{5}{14}}{\frac{20}{21}} = \frac{5/6}{20/21} = \frac{7}{8}$
Specific behaviours
✓ obtains numerator ✓ obtains denominator and simplifies

Let event A occur when no vowels are chosen in random selection of four letters from those in the word LOGARITHM.

(d) State $P(\bar{A})$. (1 mark)

Solution
$P(\bar{A}) = 1 - \frac{5}{42} = \frac{37}{42}$
Specific behaviours
✓ calculates probability

(e) Let Y be a Bernoulli random variable with parameter $p = P(A)$. Determine the mean and standard deviation of Y. (2 marks)

Solution
Y is a Bernoulli rv, so $\bar{Y} = p = \frac{5}{42} \approx 0.119$ $$\sigma_Y = \sqrt{(p(1-p)} = \sqrt{\frac{5}{42}\times\frac{37}{42}} = \frac{\sqrt{185}}{42} \approx 0.324$$
Specific behaviours
✓ indicates Bernoulli rv and states mean   ✓ states sd

(f) Determine the probability that A occurs no more than twice in ten random selections of four letters from those in the word LOGARITHM. (2 marks)

Solution
$$W \sim B\left(10, \frac{5}{42}\right)$$ $$P(W \leq 2) = 0.8933$$
Specific behaviours
✓ indicates binomial distribution with parameters   ✓ calculates probability

Question 18 **(8 marks)**

A storage container of volume 36π cm^3 is to be made in the form of a right circular cylinder with one end open. The material for the circular end costs 12c per square centimetre and for the curved side costs 9c per square centimetre.

(a) Show that the cost of materials for the container is $12\pi r^2 + \dfrac{648\pi}{r}$ cents, where r is the radius of the cylinder. (4 marks)

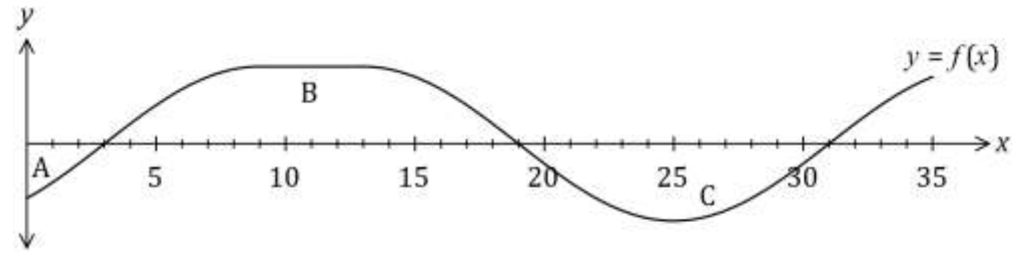
Solution
$V = \pi r^2 h \Rightarrow h = \dfrac{V}{\pi r^2}$ $h = \dfrac{36\pi}{\pi r^2} = \dfrac{36}{r^2}$ $A_{CYL} = \cancel{2}\pi r^2 + 2\pi rh$ $C = (12)(\pi r^2) + (9)(2\pi rh)$ $= 12\pi r^2 + 18\pi r \times \dfrac{36}{r^2}$ $\left(= 12\pi r^2 + \dfrac{648\pi}{r}\right)$
Specific behaviours
✓ uses volume formula ✓ expression for h in terms of r ✓ uses area formula adjusted for one end and cost ✓ substitutes for h in cost formula

(b) Use calculus techniques to determine the dimensions of the container that minimise its material costs and state this minimum cost. (4 marks)

Solution
$C'(r) = \dfrac{24\pi r^3 - 648\pi}{r^2}$ $C'(r) = 0 \Rightarrow r = 3 \text{ cm}$ $C(3) = 324\pi \text{ cents } (\$10.18)$ $h = \dfrac{36}{3^2} = 4 \text{ cm}$ Min cost of 324π cents when $r = 3$ cm and $h = 4$ cm
Specific behaviours
✓ differentiates ✓ equates $C'(r) = 0$ and solves for r ✓ determines min cost ✓ states dimensions

Question 19 **(9 marks)**

The graph of $y = f(x)$ is shown below. The areas between the curve and the $x -$ axis for regions A, B and C are 3, 20 and 12 square units respectively.



(a) Evaluate

(i) $\int_0^{31} f(x)\,dx$. (1 mark)

Solution
$\int_0^{31} f(x)\,dx = (-3) + 20 + (-12) = 5$
Specific behaviours
✓ sums signed areas

(ii) $\int_{19}^{0} f(x)\,dx$. (2 marks)

Solution
$\int_{19}^{0} f(x)\,dx = -\int_0^{19} f(x)\,dx = -\big((-3) + 20\big) = -17$
Specific behaviours
✓ reverses limits and negates ✓ sums signed areas

(iii) $\int_3^{31} 2 - 3f(x)\,dx$. (3 marks)

Solution
$\int_3^{31} 2 - 3f(x)\,dx = \int_3^{31} 2\,dx - 3\int_3^{31} f(x)\,dx = 56 - 3(8) = 32$
Specific behaviours
✓ splits integral and takes difference ✓ rectangle ✓ function

It is also known that $A(31) = 0$, where $A(x) = \int_{10}^{x} f(t)\,dt$.

(b) Evaluate

(i) $A(19)$. (1 mark)

Solution
$A(19) + \int_{19}^{31} f(t)\,dt = 0 \Rightarrow A(19) = 12$
Specific behaviours
✓ states area of region C

(ii) $A(0)$. (2 marks)

Solution
$A(3) = -(20 - 12) = -8$ $A(0) = -8 - (-3) = -5$
Specific behaviours
✓ deduces $A(3)$ ✓ deduces $A(0)$

End of questions

Additional working space

Question number: __________

Additional working space

Question number: __________